PARLIAMENT OF INDIA
(JOINT RECRUITMENT CELL)

MAIN EXAMINATION FOR POSTS OF EXECUTIVE/LEGISLATIVE/COMMITTEE/PROTOCOL OFFICER AND RESEARCH/REFERENCE OFFICER IN LOK SABHA SECRETARIAT

5th JUNE, 2009

MATHEMATICS - Paper-I

INSTRUCTIONS : Answers must be written in English only. Candidates should attempt at least 2 questions from each section and total 5 questions.

Time: 3 hours | Marks: 300

SECTION – A

Q1 - (a) Evaluate $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ (15 marks)

(b) Show that
$$x^3 - 6x^2 + 15x + 3 > 0$$
for all $x > 0$. (15 marks)

(c). If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that
$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$
(15 marks).

(d) Find all asymptotes of the curve
$$(x+2y)(x-2y)(x-y) - 2x(x-4y) + 2x = 0$$
(15 marks).

Q2- (a) Show that the vectors $(1,2,3,4)$, $(0,1,-1,2)$, $(1,5,1,8)$ and $(3,7,8,14)$ in $\mathbb{R}^4$ are linearly dependent over $\mathbb{R}$.
(15 marks).

(b). Let V be the vector space of all 2x2 matrices over $\mathbb{C}$, the field of complex numbers. Let

$$W_1 = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}, \quad W_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{C} \right\}.$$

Verify that W_1 and W_2 are subspaces of V and

$$\dim \left(\frac{W_1 + W_2}{W_2} \right) = \dim \left(\frac{W_1}{W_1 \cap W_2} \right).$$

(15 marks)

(c). If the matrix of a linear transformation on $\mathbb{R}^3$ w.r.t. the basis $\{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$ is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix},$$

what is the matrix of T w.r.t. the basis $\{(0,1,-1),\ (1,-1,1),\ (-1,1,0)\}$?

(15 marks).

(d). State Cayley - Hamilton theorem. Verify the theorem for the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

and hence find A^{-1}.

(15 marks).

Q3 - (a) Find the greatest and smallest values that the function $f(x,y) = xy$, takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

(15 marks).

(b) Evaluate

$$\iint_R e^{x^2+y^2} \, dy \, dx$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.

(15 marks).

(c) Prove that

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} \, d\theta \quad \times \quad \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi .$$

(15 marks).

(d) Test the convergence of the following improper integral :

$$\int_0^2 \frac{\log x}{\sqrt{2-x}} \, dx$$

(15 marks).

Q4- Attempt any five parts :-

(a) Show that for all real numbers x and y,

$$\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

(b) Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}}{1-e^{\frac{1}{x}}}, & \text{when } x \neq 0 \\ -1, & \text{when } x = 0 \end{cases}$$

at $x = 0$.

(c) Prove that there is no real number k for which the equation $x^5 - 45x + k = 0$ possesses two distinct roots in $[2,5]$.

(d) Find the maximum value of $\frac{\log x}{x}$ in $0 < x < \infty$.

(e). If $z = \log\left(\frac{x^3+y^3}{x+y}\right)$, prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2.$$

(f). The loop of the curve $2ay^2 = x(x-a)^2$ is revolved about x-axis. Find the volume of the solid so generated.

$(12 \times 5 = 60 \text{ marks})$

Q5- (a) Find the angle between the lines joining the origin to the points of intersection of the line $y = 3x + 2$ and the curve

$$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0.$$

(15 marks).

(b) Obtain the equation of the sphere which passes through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.

(15 marks).

(c) If O denotes the origin and α is the semi-vertical angle of a circular cone which passes through the lines Oy, Oz, $x = y = z$, then show that

$$\cos\alpha = (9 - 4\sqrt{3})^{-1/2}.$$

(15 marks).

(d) Find the equation of the circular cylinder whose guiding circle is $x^2 + y^2 + z^2 - 9 = 0$, $x - y + z = 3$.

(15 marks).

SECTION - B

Q6-(a) Solve : $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$

(15 marks).

(b) Use the method of variation of parameters to solve : $\frac{d^2y}{dx^2} + 4y = e^x$.

(15 marks).

(c) Obtain the differential equation whose solution is $Ax^2 + By^2 = 1$. What is its order and its degree ?

(15 marks).

(d) Solve : $(px - y)(py + x) = a^2 p$, where $p = \frac{dy}{dx}$

(15 marks).

Q7- (a). A particle is in equillibrium under the action of six forces. Three of these forces are reversed, and the particle remains in equillibrium. Prove that it will still remain in equillibrium if the three forces are removed altogether.

(15 marks).

(b). If D denotes $\frac{d}{dx}$, solve the following differential equation :

$$\left[x^2 D^2 - (2m-1)xD + (m^2+n^2)\right]y = n^2 x^m \log x.$$ 46

(15 marks).

(c) Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon, the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string. Prove that the tension in the string is 3W.

(15 marks).

(d). Solve : $\frac{dy}{dx} = y^2 e^{x^2/2} \log x - xy$. (15 marks)

Q8- (a) A bacteria culture is known to grow at a rate proportional to the amount present. If the initial number is 300 and if it is observed that the population has increased by 20% after 12 hours, determine the number of bacteria present in the culture after 2 days.

(15 marks).

(b) A uniform rod rests with one extremity against a rough vertical wall, the other being supported by a string of equal width fastened to a point in the wall. Prove that the least angle which the string can make with the vertical is $\tan^{-1}(3/\mu)$, μ being the coefficient of friction. (15 marks).

(c). Consider a system with two degrees of freedom for which the potential energy is given by

$$V = q_1^2 + 3q_1q_2 + 4q_2^2$$

where q_1 and q_2 are the generalized coordinates of the system. Determine the position of equillibrium and discuss its stability. (15 marks).

(d) A cable, of weight w per unit length and length $2l$ hangs from two points A and B, at the same height and at a distance $2a$ apart. Show that the maximum tension in the cable is $wa\sqrt{\frac{a}{6(l-a)}}$. (15 marks).

Q9 – (a) Determine the pitch of the wrench equipollent to two forces of magnitudes P, Q, inclined to one another at an angle α, the shortest distance between their lines of action being c.

(15 marks)

(b) A closed tube in the form of an equilateral triangle contains equal volumes of three liquids which do not mix, and is placed with its lowest side horizontal. Prove that, if the densities of the liquids are in arithmetic progression, their surface of separation will be at points of trisection of the sides of the triangle. (15 marks)

(c) Show that the depth of the centre of pressure of a trapezium, of which one side of length 'a' is in the surface and the parallel side of length 'b' is at a depth h, is $\left(\frac{a+3b}{a+2b}\right)\frac{h}{2}$, neglecting the pressure of the atmosphere. (15 marks).

(d). Find constants α, β, γ so that $\vec{v} = (x+2y+\alpha z)\hat{i} + (\beta x - 3y - z)\hat{j} + (4x + \gamma y + 2z)\hat{k}$ is irrotational. Also show that $\vec{v}$ can be expressed as gradient of a scalar function. (15 marks).

Q10 - (a) Evaluate by Green's theorem

$$\int_C (\cos x \sin y - xy)\, dx + \sin x \cos y \, dy,$$

where C is the circle $x^2+y^2=1$. (15 marks)

(b) (i) If $\vec{A}$ and $\vec{B}$ are irrotational, show that $\vec{A} \times \vec{B}$ is solenoidal.

(ii) Prove that $\text{curl}(\phi \, \text{grad}\phi) = \vec{0}$. (15 marks)

(c) Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has maximum of magnitude 64 in a direction parallel to the z-axis. (15 marks)

(d). State Serret-Frenet's formulae. Calculate the curvature and torsion of the curve $\vec{r} = \vec{r}(u)$ in terms of u, where $\vec{r}(u) = (u, u^2, u^3)$.

(15 marks)

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5th JUNE, 2009

MATHEMATICS - Paper-II

INSTRUCTIONS : Answers must be written in English only. Candidates should attempt at least 2 questions from each section and total 5 questions. All questions carry equal marks. All symbols have their usual meaning.

Time: 3 hours | Marks: 300

SECTION – A

1 (a) Let G be the group generated by

$$\{x, y \mid x^2 = e, y^4 = e, xy = y^{-1}x\}$$

Let $H = \langle x \rangle$ and $K = \langle y \rangle$.

Then show that $K \trianglelefteq G$, $H \ntrianglelefteq G$ and $G = HK$

Give a geomentric interpretation of the generators of G.

(b) Let G be a group such that $o(G) = pq$, where p, q are primes such that $p > q$ and $q \nmid p-1$.

Prove that G is cyclic

If $q \mid p-1$, does $o(G) = pq$ imply that G is cyclic? comment on it.

(c) Consider the ring $C[0,1]$, the class of all continuous real valued functions defined on the closed interval $[0,1]$. Let $M = \{f \in C[0,1] \mid f(1/2) = 0\}$. Show that

M is an ideal of $C[0,1]$.
Find the principal ideals of Z_{10} generated by $\bar{3}$ and $\bar{5}$.

2. (a) Prove that a subset of $\mathbb{R}$ is compact iff it is closed and bounded. Show that open intervals on $\mathbb{R}$ are not compact

(b) If $\lim_{n\to\infty} a_n = l$, where $a_n > 0$, for all n, then show that $\lim_{n\to\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} = l$.

Hence or otherwise show that if $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$,

then $\lim_{n\to\infty} \left[\left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n} = e$

(c) Let ϕ be bounded and monotonic in $[a,\infty)$ and $\lim_{x\to\infty} \phi(x) \to 0$. Let $\int_a^X f\,dx$ be bounded when $X \geq a$. Then prove that $\int_a^\infty f\phi\,dx$ is convergent at ∞.

Hence discusss the convergence of

$$\int_a^\infty e^{-ax} \frac{\sin x}{x}\,dx, \quad a \geq 0$$

3. (a) If a series $\sum f_n$ converges uniformly to f on $[a,b]$ and its terms f_n are continuous at x_0 of $[a,b]$, then prove that the sum function is also continuous at x_0. Test

the uniform convergence of the series $\sum x e^{-nx}$ and continuity of its sum function at $x=0$.

(b) Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ denote the unit sphere in $\mathbb{R}^3$. Evaluate the surface integral over S:

$$\iint_S (x^2+y+z)\, dA.$$

4 (a) Show that for the function

$$f(z) = \begin{cases} (\bar{z})^2/z, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the Cauchy Riemann equations are satisfied at the origin. Does $f'(0)$ exist?

(b) Evaluate the Integral $I = \oint_C \frac{3z-1}{z^3-z}\, dz$, where C is any rectangle containing the points $z=0$ and $z=\pm 1$.

(c) Classify the singular points of the function

$$f(z) = \frac{(1-e^z)\cos(1/(z-8))}{z^2(z-8)(z^2-16)}$$

in the finite complex plane

5 (a) Prove that if the i^{th} constraint in the primal is equality, then the j^{th} dual variable is unrestricted in sign.

(b) Let $X_B = B^{-1}b$ be a basic feasible solution of the linear programming problem Max $Z = CX$ such that $AX = b$, $X \geq 0$ and $z^* = C_B X_B$, is the value of the objective function for this basic feasible solution. If $c_j - z_j \leq 0$ for every column a_j in A but not in B, then prove that z^* is the optimal solution of the objective function z.

(c) Solve using Simplex method:

$$\text{Max } z = 3x_1 + 5x_2 + 4x_3$$

$$\text{subject to} \quad 2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

$$\text{and } x_1, x_2, x_3 \geq 0$$

SECTION - B

6 (a) Find the complete solution of the equation

$$xp + 3yq = 2(z^2 - x^2q^2)$$

(b) A string is fixed at two points l apart and is stretched. The motion takes place by displacing the string in the form $y = a\sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x frome one end at time t is

$$y(x,t) = a\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi c t}{l}\right)$$

7 (a)(i) Given a set of $(n+1)$ distinct data points (x_i, y_i) $(i = p, p+1, \dots p+n)$ with $y_i = f(x_i)$, show that the n^{th} divided difference of f relative to $x_p, x_{p+1}, \dots, x_{p+n}$ may be expressed as

$$f[x_p, x_{p+1}, \dots; x_{p+n}] = \sum_{i=p}^{p+n}\left[\frac{y_i}{\prod_{\substack{j=p \\ j\neq i}}^{p+n}(x_i - x_j)}\right]$$

(ii) Construct interpolating polynomial using the following data:

0	1	2
1	0.135335	0.018316

and hence find $f(1.2)$

(b) Solve the following system of linear equations using decomposition method (i) without partial pivoting and (ii) with partial pivoting and hence comment on the obtained results regarding accuracy.

$$x_1 + 7x_2 + 6x_3 = 2$$
$$x_1 - 2x_2 - x_3 = 6$$
$$2x_1 + 2x_2 - 4x_3 = 8$$

8 (a) If $y = A + Bx + Cx^2$ and y_0, y_1, y_2 are the values of y corresponding to $x = a, a+h, a+2h$ respectively, prove that

$$\int_a^{a+2h} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Use it to obtain an approximate value of π from the integral $\int_0^1 \frac{dx}{1+x^2}$

(b) Using Runge-Kutta fourth order method, evaluate $y(1.1)$ from the following differential equation:

$$\frac{dy}{dx} = x^2 + y^2, \quad y(1) = 0.$$